Journal of the Egyptian Mathematical Society (2015) 23, 139–143

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Fuzzy cyclic contraction and fixed point theorems

Satish Shukla [a], Sunny Chauhan [b,*]

[a] Department of Applied Mathematics, Shri Vaishnav Institute of Technology & Science, Gram Baroli, Sanwer Road, Indore, MP 453331, India
[b] Near Nehru Training Centre, H. No. 274, Nai Basti B-14, Bijnor 246701, Uttar Pradesh, India

Received 21 November 2013; revised 31 January 2014; accepted 6 February 2014
Available online 15 March 2014

KEYWORDS
Fuzzy cyclic contraction;
Fuzzy metric space;
Fixed point

Abstract In this paper, we consider some fuzzy cyclic contractions and prove the existence and uniqueness of fixed points of operators belong to a class consisting fuzzy cyclic operators defined on a subset of a fuzzy metric space. We also furnish some illustrative examples to support our main results.

2010 MATHEMATICS SUBJECT CLASSIFICATION: Primary 47H10; Secondary 54H25

1. Introduction and preliminaries

In 1965, Zadeh [1] introduced the concept of fuzzy sets, as a new way to represent the vagueness in every day life. Kramosil and Michalek [2] investigated the notion of fuzzy metric space which is closely related to a class of probabilistic metric spaces. In [3,4], George and Veeramani modified the concept of fuzzy metric space of Kramosil and Michalek, and obtained a Hausdorff and first countable topology on the modified fuzzy metric space. They also obtained a Hausdorff topology for this kind of fuzzy metric space which has very important applications in quantum particle physics, particularly in connection with both string and ϵ^{∞} theory (see, [5] and references therein). Since then, Gregori and Romaguera [6] proved that the topology induced by a fuzzy metric space in George and Veeramani's sense is metrizable. Grabiec [7] obtained a fuzzy version of the Banach contraction principle in fuzzy metric spaces in Kramosil and Michalek's sense. Many mathematicians proved several fixed point results in fuzzy metric spaces (see, for instance [8–18]).

Gregori and Sapena [19] introduced the concept of fuzzy contractive mappings and proved some fixed point results for such mappings. Kirk et al. [20] introduced the notion of a cyclic representation and characterized the Banach contraction principle in the context of a cyclic mapping. Some interesting fixed point results for cyclic contraction in fuzzy metric spaces can be seen in [21–23]. In this paper, we generalize and extend the concept of fuzzy contractive mappings to fuzzy cyclic contraction and prove some fixed point results for operators belong to a class consisting fuzzy cyclic operators in complete fuzzy metric spaces. Some examples are provided which illustrate the results.

For the sake of completeness, we recall some definitions and properties of fuzzy metric spaces.

* Corresponding author. Tel.: +91 9412452300.
E-mail addresses: satishmathematics@yahoo.co.in (S. Shukla), sun.gkv@gmail.com (S. Chauhan).

Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

1110-256X © 2014 Production and hosting by Elsevier B.V. on behalf of Egyptian Mathematical Society.
http://dx.doi.org/10.1016/j.joems.2014.02.008

Definition 1.1 [1]. A fuzzy set A in a nonempty set X is a function with domain X and values in $[0,1]$.

Definition 1.2 [24]. A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $\{[0,1], \star\}$ is an abelian topological monoid with unit 1 such that $a \star b \leqslant c \star d$ whenever $a \leqslant c$ and $b \leqslant d$, $a, b, c, d \in [0,1]$. Three typical examples of t-norms are $a \star b = \min\{a, b\}$ (minimum t-norm), $a \star b = ab$ (product t-norm), and $a \star b = \max\{a + b - 1, 0\}$ (Lukasiewicz t-norm).

Definition 1.3 [3]. The triplet $(X, M, \star)$ is called a fuzzy metric space if X is an arbitrary set, $\star$ is a continuous t-norm and M is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions: (for all $x, y, z \in X$ and $s, t > 0$)

(M1) $M(x, y, t) > 0$;
(M2) $M(x, y, t) = 1$ if and only if $x = y$;
(M3) $M(x, y, t) = M(y, x, t)$;
(M4) $M(x, y, t) \star M(y, z, s) \leqslant M(x, z, t + s)$;
(M5) $M(x, y, \cdot) : (0, \infty) \to [0,1]$ is continuous.

Here M with $\star$ is called a fuzzy metric on X. Note that, $M(x, y, t)$ can be thought of as the definition of nearness between x and y with respect to t. It is known that $M(x, y, \cdot)$ is nondecreasing for all $x, y \in X$ (see [4]). For examples of fuzzy metric spaces we refer to [25].

Let $(X, M, \star)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : \ M(x, y, t) > 1 - r\}.$$

The collection $\{B(x, y, t) : \ x \in X, \ 0 < r < 1, t > 0\}$ is a neighborhood system for a topology τ on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

Lemma 1.1 (*[4,26]*). *Let $(X, M, \star)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.*

Definition 1.4 [3]. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \star)$ is said to be convergent to $x \in X$ if for each $\varepsilon \in (0,1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n > n_0$.

Definition 1.5 [3]. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \star)$ is said to be a Cauchy sequence if for each $\varepsilon \in (0,1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m > n_0$.

A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

Definition 1.6 [7]. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \star)$ is called G-Cauchy if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1,$$

for each $t > 0$ and $p > 0$.

The above definitions of Cauchy sequences are different; for details we refer to [27].

Theorem 1.1 [3]. *A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \star)$ converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$.*

It is well known that every closed subset of a complete fuzzy metric space is complete.

In [19], Gregori and Sapena introduced the concept of fuzzy contractive mappings as follows:

Let $(X, M, \star)$ be a fuzzy metric space. We say that the mapping $T : X \to X$ is fuzzy contractive if there exists $k \in [0,1)$ such that

$$\frac{1}{M(Tx, Ty, t)} - 1 \leqslant k\left[\frac{1}{M(x, y, t)} - 1\right] \tag{1.1}$$

for all $x, y \in X$ and $t > 0$. Here k is called the fuzzy contractive constant of T.

2. Main results

In this section we define a class of cyclic operators on fuzzy metric spaces and investigate the existence and uniqueness of fixed points of these operators.

In [20], the following concept of cyclic representation of a set is defined.

Let X be a nonempty set and $T : X \to X$ be an operator, $A_1, A_2, \ldots, A_m$ be subsets of X. Then $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T if

(a) A_i, $i = 1, 2, \ldots, m$ are nonempty sets;
(b) $T(A_1) \subset A_2, \ldots, T(A_{m-1}) \subset A_m$, $T(A_m) \subset A_1$.

Definition 2.1. Let $(X, M, \star)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be subsets of X and $Y = \bigcup_{i=1}^m A_i$. An operator $T : Y \to Y$ is called fuzzy cyclic contraction if the following conditions hold:

(i) $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;
(ii) there exists $k \in [0,1)$ such that

$$\frac{1}{M(Tx, Ty, t)} - 1 \leqslant k\left[\frac{1}{M(x, y, t)} - 1\right] \tag{2.1}$$

for any $x \in A_i$, $y \in A_{i+1}$ $(i = 1, 2, \ldots, m$, where $A_{m+1} = A_1)$ and each $t > 0$.

Definition 2.2. Let $(X, M, \star)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be subsets of X and $Y = \bigcup_{i=1}^m A_i$. A self operator T of Y is said to belong to the class $D_M^C(k_1, k_2, k_3)$ if the following conditions hold:

(i) $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;
(ii) there exist $0 \leqslant k_1, k_2, k_3 < 1$ such that

$$\begin{aligned}\frac{1}{M(Tx, Ty, t)} - 1 \leqslant{}& k_1\left[\frac{1}{M(x, y, t)} - 1\right] \\ &+ k_2\left[\frac{1}{M(x, Tx, t)} - 1\right] \\ &+ k_3\left[\frac{1}{M(y, Ty, t)} - 1\right]\end{aligned} \tag{2.2}$$

for any $x \in A_i,\ y \in A_{i+1}$ ($i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$) and each $t > 0$.

It is obvious that if an operator T is in the class $D_M^C(k, 0, 0)$ with $0 \leqslant k < 1$, then T is a fuzzy cyclic contraction.

Now we investigate the existence of fixed points of operators in the class $D_M^C(k_1, k_2, k_3)$. The first proposition gives uniqueness conditions of the fixed point of an operator provided that the fixed point exists.

Proposition 2.1. *Let $(X, M, \star)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be subsets of X and $Y = \bigcup_{i=1}^m A_i$. Let T be a self operator of Y and belongs to $D_M^C(k_1, k_2, k_3)$. If $\mathcal{F}(T) = \{x \in Y :\ Tx = x\} \neq \emptyset$, then $\mathcal{F}(T)$ consists of a single point.*

Proof. Assume the contrary, that $u, v \in \mathcal{F}(T) \subset Y = \bigcup_{i=1}^m A_i,\ u \neq v$. Note that, $u \in A_i$ for some $1 \leqslant i \leqslant m$, so $u = Tu \in A_{i+1}$ and so on. Hence, $u \in \bigcap_{i=1}^m A_i$ and similar result holds for v therefore, it follows from (2.2) that

$$\begin{aligned}\frac{1}{M(u,v,t)} - 1 &= \frac{1}{M(Tu,Tv,t)} - 1\\ &\leqslant k_1\left[\frac{1}{M(u,v,t)} - 1\right] + k_2\left[\frac{1}{M(u,Tu,t)} - 1\right]\\ &\quad + k_3\left[\frac{1}{M(v,Tv,t)} - 1\right]\\ &\leqslant k_1\left[\frac{1}{M(u,v,t)} - 1\right] < \frac{1}{M(u,v,t)} - 1,\end{aligned}$$

a contradiction. Therefore, we must have $u = v$. □

We now define asymptotically regular operators in fuzzy metric spaces.

Definition 2.3. Let $(X, M, \star)$ be any fuzzy metric space, T be a self mapping of X and $x \in X$. The mapping T is said to be asymptotically regular at point x if

$$\lim_{n\to\infty} M(T^n x, T^{n+1}x, t) = 1 \quad \text{for all } t > 0.$$

The next result gives a condition for a cyclic operator to be asymptotically regular.

Proposition 2.2. *Let $(X, M, \star)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be subsets of X and $Y = \bigcup_{i=1}^m A_i$. Let T be a self operator of Y and belongs to $D_M^C(k_1, k_2, k_3)$ with $k_1 + k_2 + k_3 < 1$. Then T is asymptotically regular at every point $x \in Y$.*

Proof. Take an arbitrary point $x_0 \in Y$ and define the sequence of Picard's iterates $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. As, $x_0 \in Y = \bigcup_{i=1}^m A_i$, so for all $n \in \mathbb{N}$ there exists i such that $1 \leqslant i \leqslant m$ and $x_n \in A_i$ and so $x_{n+1} = Tx_n \in A_{i+1}$. Therefore for all $t > 0$, it follows from (2.2) that

$$\begin{aligned}\frac{1}{M(x_{n+1},x_{n+2},t)} - 1 &= \frac{1}{M(Tx_n,Tx_{n+1},t)} - 1\\ &\leqslant k_1\left[\frac{1}{M(x_n,x_{n+1},t)} - 1\right] + k_2\left[\frac{1}{M(x_n,Tx_n,t)} - 1\right]\\ &\quad + k_3\left[\frac{1}{M(x_{n+1},Tx_{n+1},t)} - 1\right] = k_1\left[\frac{1}{M(x_n,x_{n+1},t)} - 1\right]\\ &\quad + k_2\left[\frac{1}{M(x_n,x_{n+1},t)} - 1\right] \ \ + k_3\left[\frac{1}{M(x_{n+1},x_{n+2},t)} - 1\right],\end{aligned}$$

that is,

$$\begin{aligned}(1-k_3)\left[\frac{1}{M(x_{n+1},x_{n+2},t)} - 1\right] &\leqslant (k_1+k_2)\left[\frac{1}{M(x_n,x_{n+1},t)} - 1\right]\\ \frac{1}{M(x_{n+1},x_{n+2},t)} - 1 &\leqslant \frac{k_1+k_2}{1-k_3}\left[\frac{1}{M(x_n,x_{n+1},t)} - 1\right].\end{aligned}$$

By repetition of this process we obtain

$$\frac{1}{M(x_{n+1},x_{n+2},t)} - 1 \leqslant \left(\frac{k_1+k_2}{1-k_3}\right)^{n+1}\left[\frac{1}{M(x_0,x_1,t)} - 1\right]. \tag{2.3}$$

Since $k_1 + k_2 + k_3 < 1$, so $\frac{k_1+k_2}{1-k_3} < 1$ and hence, $\lim_{n\to\infty} M(x_{n+1}, x_{n+2}, t) = 1$. Therefore, for any arbitrary $x_0 \in Y$ we have $\lim_{n\to\infty} M(T^{n+1}x_0, T^{n+2}x_0, t) = 1$ for all $t > 0$, and result follows. □

We now prove the existence of fixed point of operators in the class $D_M^C(k_1, k_2, k_3)$.

Theorem 2.1. *Let $(X, M, \star)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be subsets of X and $Y = \bigcup_{i=1}^m A_i$ is such that every G-Cauchy sequence in Y is convergent in Y. Let T be a self operator of Y and belongs to $D_M^C(k_1, k_2, k_3)$ with $k_1 + k_2 + k_3 < 1$. Then T has a unique fixed point $u \in Y$ and the sequence of Picard's iterates $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$, where $x_0 \in Y$ is arbitrary, converges to the fixed point of T.*

Proof. Let $n \in \mathbb{N}$ then for any $p \geqslant 1$ we have

$$\begin{aligned}M(x_n,x_{n+p},t) &\geqslant M(x_n,x_{n+1},t/2) \star M(x_{n+1},x_{n+p},t/2)\\ &\geqslant M(x_n,x_{n+1},t/2) \star M(x_{n+1},x_{n+2},t/2^2)\\ &\quad \star M(x_{n+2},x_{n+p},t/2^2)\\ &\geqslant M(x_n,x_{n+1},t/2) \star M(x_{n+1},x_{n+2},t/2^2)\\ &\quad \star M(x_{n+2},x_{n+3},t/2^3)\\ &\quad\quad \star \cdots \star M(x_{n+p-1},x_{n+p},t/2^{p-1}).\end{aligned} \tag{2.4}$$

Setting $\lambda = \frac{k_1+k_2}{1-k_3}$ and $M_n(t) = M(x_n, x_{n+1}, t)$ for all $t > 0$ and $n \geqslant 0$, it follows from inequality (2.3) of Proposition 2.2 that

$$\frac{1}{M_{n+1}(t)} \leqslant \frac{\lambda^{n+1}}{M_0(t)} + 1 - \lambda^{n+1} \leqslant \frac{\lambda^{n+1}}{M_0(t)} + 1,$$

that is,

$$\frac{1}{\frac{\lambda^{n+1}}{M_0(t)} + 1} \leqslant M_{n+1}(t) \quad \text{for all } t > 0 \text{ and } n \geqslant 0.$$

Using the above inequality in (2.4) we obtain

$$\begin{aligned}M(x_n,x_{n+p},t) &\geqslant M_n(t/2) \star M_{n+1}(t/2^2) \star M_{n+2}(t/2^3)\\ &\quad \star \cdots \star M_{n+p-1}(t/2^{p-1})\\ &\geqslant \frac{1}{\frac{\lambda^n}{M_0(t/2)} + 1} \star \frac{1}{\frac{\lambda^{n+1}}{M_0(t/2^2)} + 1} \star \cdots \star \frac{1}{\frac{\lambda^{n+p-1}}{M_0(t/2^{p-1})} + 1}\\ &\geqslant \frac{1}{\frac{\lambda^n}{M_0(t/2)} + 1} \star \frac{1}{\frac{\lambda^n}{M_0(t/2^2)} + 1} \star \cdots \star \frac{1}{\frac{\lambda^n}{M_0(t/2^{p-1})} + 1}.\end{aligned}$$

As, $\lambda < 1$, letting $n \to \infty$, we obtain from the above inequality that

$$\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1 \quad \text{for all } t > 0, p \geqslant 1. \tag{2.5}$$

Thus, $\{x_n\}$ is a G-Cauchy sequence in Y therefore by the assumption there exists $u \in Y$ such that

$$\lim_{n\to\infty} M(x_n, u, t) = 1 \quad \text{for all } t > 0. \tag{2.6}$$

We shall show that u is the fixed point point of T.

Note that, As $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T, the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \ldots, m\}$. Suppose that $u \in A_i$ then we have $Tu \in A_{i+1}$, also take a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \in A_{i+1}$. Then, for any $t > 0$ we have

$$\begin{aligned}
\frac{1}{M(x_{n_k+1}, Tu, t)} - 1 &= \frac{1}{M(Tx_{n_k}, Tu, t)} - 1 \\
&\leqslant k_1\left[\frac{1}{M(x_{n_k}, u, t)} - 1\right] \\
&\quad + k_2\left[\frac{1}{M(x_{n_k}, Tx_{n_k}, t)} - 1\right] \\
&\quad + k_3\left[\frac{1}{M(u, Tu, t)} - 1\right] \\
&\leqslant k_1\left[\frac{1}{M(x_{n_k}, u, t)} - 1\right] \\
&\quad + k_2\left[\frac{1}{M(x_{n_k}, x_{n_k+1}, t)} - 1\right] \\
&\quad + k_3\left[\frac{1}{M(u, Tu, t)} - 1\right].
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.5) and (2.6), we obtain

$$\frac{1}{M(u, Tu, t)} - 1 \leqslant k_3\left[\frac{1}{M(u, Tu, t)} - 1\right].$$

As, $0 \leqslant k_3 < 1$ we must have $\frac{1}{M(u,Tu,t)} - 1 = 0$, that is, $M(u, Tu, t) = 1$ for all $t > 0$, hence $Tu = u$. Thus, u is a fixed point of T. Therefore, $\mathcal{F}(T) = \{x \in Y : Tx = x\} \neq \emptyset$, and then by Proposition 2.1, $\mathcal{F}(T)$ consists of a single point, that is, the fixed point of T is unique. ☐

Following corollaries are immediate consequence of the above theorem.

Corollary 2.1. *Let $(X, M, \bigstar)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be subsets of X and $Y = \bigcup_{i=1}^{m} A_i$ is such that every G-Cauchy sequence in Y is convergent in Y. Let T be a fuzzy cyclic contraction on Y. Then T has a unique fixed point $u \in Y$ and the sequence of Picard's iterates $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$, where $x_0 \in Y$ is arbitrary, converges to the fixed point of T.*

Corollary 2.2. *Let $(X, M, \bigstar)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be subsets of X and $Y = \bigcup_{i=1}^{m} A_i$ is such that every G-Cauchy sequence in Y is convergent in Y. Let T be a self operator of Y such that the following conditions hold:*

(i) *$Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;*
(ii) *there exist $0 \leqslant k_1, k_2 < 1$ such that $k_1 + k_2 < 1$ and*

$$\frac{1}{M(Tx, Ty, t)} - 1 \leqslant k_1\left[\frac{1}{M(x, Tx, t)} - 1\right] + k_2\left[\frac{1}{M(y, Ty, t)} - 1\right] \tag{2.7}$$

for any $x \in A_i$, $y \in A_{i+1}$ ($i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$) and each $t > 0$.

Then T has a unique fixed point $u \in Y$ and the sequence of Picard's iterates $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$, where $x_0 \in Y$ is arbitrary, converges to the fixed point of T.

Next we give some examples which illustrate the above results.

Example 2.1. Let $X = [0, 1]$ and $\bigstar$ be the usual product. Consider the fuzzy set $M: X^2 \times (0, \infty) \to [0, 1]$ defined by

$$M(x, y, t) = \frac{t}{t + |x - y|} \quad \text{for all } x, y \in X \text{ and } t > 0.$$

Then $(X, M, \bigstar)$ is a complete fuzzy metric space. Let, $A_1 = \left[0, \frac{2}{5}\right]$, $A_2 = \left[\frac{1}{5}, 1\right]$ and $Y = A_1 \bigcup A_2$. Define $T: Y \to Y$ by

$$Tx = \begin{cases} \frac{77}{120}, & \text{if } x \in [0, 1/2]; \\ \frac{11}{25}(1 - x), & \text{if } x \in (1/2, 1]. \end{cases}$$

Then, it is clear that $Y = A_1 \bigcup A_2$ is a cyclic representation of Y with respect to T.

We shall show that T is in the class $D_M^C\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{6}\right)$. The case $x \in \left[0, \frac{2}{5}\right]$ and $y \in \left[\frac{1}{5}, \frac{1}{2}\right]$ is trivial.

If, $x \in \left[0, \frac{2}{5}\right]$ and $y \in \left(\frac{1}{2}, 1\right]$ then

$$\begin{aligned}
\frac{1}{M(Tx, Ty, t)} - 1 &= \frac{1}{M\left(\frac{77}{120}, \frac{11}{25}(1 - y), t\right)} - 1 = \frac{t + \left|\frac{11}{25}y - \frac{121}{600}\right|}{t} - 1 \\
&= \frac{\left|\frac{1}{5}y - \frac{1}{5}x + \frac{1}{5}x - \frac{77}{600} + \frac{6}{25}y - \frac{11}{150}\right|}{t} \\
&\leqslant \frac{\frac{1}{5}|y - x| + \frac{1}{5}\left|x - \frac{77}{120}\right| + \frac{1}{6}\left|\frac{36}{25}y - \frac{11}{25}\right|}{t} \\
&= \frac{1}{5}\left[\frac{t + |x - y|}{t} - 1\right] + \frac{1}{5}\left[\frac{t + |x - Tx|}{t} - 1\right] \\
&\quad + \frac{1}{6}\left[\frac{t + |y - Ty|}{t} - 1\right],
\end{aligned}$$

that is,

$$\frac{1}{M(Tx, Ty, t)} - 1 \leqslant \frac{1}{5}\left[\frac{1}{M(x, y, t)} - 1\right] + \frac{1}{5}\left[\frac{1}{M(x, Tx, t)} - 1\right] + \frac{1}{6}\left[\frac{1}{M(y, Ty, t)} - 1\right].$$

Therefore, T is in the class $D_M^C\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{6}\right)$. Note that, all the conditions of Theorem 2.1 are satisfied and $u = \frac{11}{36} \in A_1 \cap A_2$ is the unique fixed point of T.

Example 2.2. Let $X = \{1, 2, 3, 4, 5\}$ and $\bigstar$ be the usual product. Consider the fuzzy set $M: X^2 \times (0, \infty) \to [0, 1]$ defined by

$$M(x, y, t) = \begin{cases} \frac{x}{y}, & \text{if } x \leqslant y; \\ \frac{y}{x}, & \text{if } y \leqslant x. \end{cases} \text{ for all } x, y \in X \text{ and } t > 0.$$

Then $(X, M, \bigstar)$ is a complete fuzzy metric space. Let, $A_1 = \{1, 2, 3\}$, $A_2 = \{3, 5\}$ and $Y = A_1 \bigcup A_2$. Define, $T: Y \to Y$ by

$T1 = 3, \quad T2 = 5, \quad T3 = 3, \quad T4 = 3.$

Then, it is clear that $Y = A_1 \bigcup A_2$ is a cyclic representation of Y with respect to T. Now by a careful calculation one can see that $T \in D_M^C\left(0, \frac{4}{9}, \frac{4}{9}\right)$.

Note that, all the conditions of Corollary 2.2 are satisfied and $u = 3 \in A_1 \bigcap A_2$ is the unique fixed point of T. Also, for points $x = 2,\ y = 3$ one can see that there is no $k \in [0, 1)$ such that (2.1) is satisfied. Therefore, T is not a fuzzy cyclic contraction.

Acknowledgements

The authors are thankful to the anonymous referee for his valuable suggestions. We are also very grateful to Professor Abdel-shafi Obada and Professor Ahmed M. El Sayed for their kind support.

References

[1] L.A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.
[2] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 15 (1975) 326–334.
[3] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (1994) 395–399.
[4] A. George, P. Veeramani, On some results of analysis for fuzzy metric spaces, Fuzzy Sets Syst. 90 (1997) 365–368.
[5] M.S. El Naschie, On a fuzzy Khaler-like manifold which is consistent with two slit experiment, Int. J. Nonlinear Sci. Numer. Simul. 6 (2005) 95–98.
[6] V. Gregori, S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets Syst. 115 (2000) 485–489.
[7] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets Syst. 27 (1988) 385–389.
[8] I. Altun, Some fixed point theorems for single and multivalued mappings on ordered Non-Archimedean fuzzy metric spaces, Iranian J. Fuzzy Syst. 7 (3) (2010) 91–96.
[9] Z.K. Deng, Fuzzy pseudo-metric spaces, J. Math. Anal. Appl. 86 (1982) 74–95.
[10] F. Merghadi, A. Aliouche, A related fixed point theorem in N-fuzzy metric spaces, Iranian J. Fuzzy Syst. 7 (3) (2010) 73–86.
[11] R. Saadati, S. Sedghi, H. Zhou, A common fixed point theorem for weakly commuting maps in $\mathcal{L}$-fuzzy metric spaces, Iranian J. Fuzzy Syst. 5 (1) (2008) 47–53.
[12] S. Sedghi, K.P.R. Rao, N. Shobe, A common fixed point theorem for six weakly compatible maps in $\mathcal{M}$-fuzzy metric spaces, Iranian J. Fuzzy Syst. 5 (2) (2008) 49–62.
[13] W. Sintunavarat, P. Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math. 2011 (2011) 14 (Article ID 637958).
[14] W. Sintunavarat, P. Kumam, Common fixed points for R-weakly commuting in fuzzy metric spaces, Ann. dell'Univer. Ferrara 58 (2) (2012) 389–406.
[15] M.A. Ahmed, Fixed point theorems in fuzzy metric spaces, J. Egypt. Math. Soc. 22 (1) (2014) 59–62.
[16] H.K. Nashine, Z. Kadelburg, S. Radenović, Fixed point theorems via various cyclic contractive conditions in partial metric spaces, Publications de L'Institut Mathématique, Nouvelle série, tome 93 (107) (2013) 69–93.
[17] S. Chauhan, S. Radenović, M. Imdad, C. Vetro, Some integral type fixed point theorems in Non-Archimedean Menger PM-Spaces with common property (E.A) and application of functional equations in dynamic programing, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales, Serie A. Mathematics (RASCAM). http://dx.doi.org/10.1007/s13398-013-0142-6.
[18] S. Shukla, Fuzzy $\mathcal{H}$-weak contractions and fixed point theorems in fuzzy metric spaces, Gulf J. Math. 2 (2013) 67–75.
[19] V. Gregori, A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets Syst. 125 (2002) 245–252.
[20] W.A. Kirk, P.S. Srinivasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theor. 4 (1) (2003) 79–89.
[21] D. Gopal, M. Imdad, C. Vetro, M. Hasan, Fixed point theory for cyclic weak ϕ-contraction in fuzzy metric spaces, J. Nonlinear Anal. Appl. (2012) 11 (Article ID jnaa-00110).
[22] Y. Shen, W. Chen, Fixed point theorems for cyclic contraction mappings in fuzzy metric spaces, Fixed Point Theor. Appl. 2013 (2013) 133.
[23] Y. Shen, D. Qiu, W. Chen, Fixed point theory for cyclic φ-contractions in fuzzy metric spaces, Iranian J. Fuzzy Syst. 10 (4) (2013) 125–133.
[24] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960) 313–334.
[25] A. Sapena, A contribution to the study of fuzzy metric spaces, Appl. Gen. Topol. 2 (1) (2001) 63–75.
[26] J.R. López, S. Romaguera, The Hausdorff fuzzy metric on compact sets, Fuzzy Sets Syst. 147 (2) (2004) 273–283.
[27] R. Vasuki, P. Veeramani, Fixed point theorems and Cauchy sequences in fuzzy metric spaces, Fuzzy Sets Syst. 135 (2003) 415–417.